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LETTER TO THE EDITOR

Flory calculation of the fractal dimensionality of the shortest path in a percolation cluster

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Abstract. A Flory-type calculation of the fractal dimensionality of the shortest path in a percolation cluster near the threshold is introduced. The result is roughly consistent with previous predictions or numerical simulations.

The percolation theory which originally dealt only with scalar transport properties (i.e. heat or fluid flow, electrical conduction, etc) has now come into the field of mechanical behaviour [1, 2, 3]. Some concepts are being proposed which aim to describe the relevant 'objects' which govern the mechanical properties of random structures. Among these, some are classical (e.g. sensible or singly connected bonds) while others are new. Herrmann *et al* [4] introduced one such object: the elastic backbone (the union of the *shortest paths* linking two points within the percolation cluster). A deterministic fractal model of the backbone [1]—a Sierpinski gasket—although probably too simple, also emphasises the dominant role of the shortest paths. In this letter a Flory derivation of the dimensionality of these paths is given. Another area of interest is the problem of the first passage [5] which is highly relevant in some 'percolation' models of forest fires or of disease propagation, etc.

Flory calculation is known to give excellent values of fractal dimensionality for a self-avoiding walk [6]† or for a percolation cluster [7], even though the mean-field approximations used therein are valid only for large space dimensions [6].

The chemical length [13] of the shortest path (number of distinct sites L) between two points separted by an Euclidian distance R can be written as $L \sim R^{\alpha}$. The exponent α can be related simply to the spreading [8] or connectivity [9] dimension which gives the number M of distinct sites which can be reached after N steps:

$$M \sim N^{\hat{d}}.$$

Expressing that the mass contained in a sphere of radius R scales as $M \sim R^{D}$ where D is the fractal dimension of the infinite cluster, the relation $\alpha = D/\hat{d}$ is obtained. As noted by various authors [9, 10], $\hat{d}$ is an intrinsic topological dimension (related to the connectedness of the network) where D is a geometrical one and is a function of the embedding space.

Two evident statements on the shortest path between two given points in an infinite cluster will allow an estimation of the elastic energy; (i) the shortest path is a

† De Gennes (1979) has shown that, in some cases, two errors made in the Flory calculation roughly cancel each other.

self-avoiding walk (no loop is possible) and (ii) the shortest path 'avoids' the rest of the cluster. The shortest path is then modelled by a random walk (which will give the entropy term) *modified* by the interaction with the whole cluster (i.e. itself + the rest of the cluster) (energy term). Therefore, the entropy is proportional to R^2/L (deliberately omitting all prefactors) and the energy to LR^{D-d} (the number of elements in the walk times the mean density of obstacles). The most probable configuration is obtained by minimising the expression

$$R^2/L + LR^{D+d}.$$

This leads to an exponent $\alpha (L \sim R^\alpha)$:

$$\alpha = \tfrac{1}{2}(d+2-D) \qquad \text{for } d \leqslant 6$$
$$2 \qquad \text{for } d \geqslant 6.$$

The result $\alpha = 2$ is known to be exact for a Bethe lattice, which is a good description of the percolation cluster for $d \geqslant 6$. Previous estimates of α are given in table 1. If a Flory-calculated value [7] is used for $D \simeq \tfrac{1}{2}(d+2)$, α takes a very simple expression (although less accurate):

$$\alpha \simeq \tfrac{1}{4}(d+2) \qquad \text{for } d < 6.$$

Table 1. Estimates of α for given values of α and D

d	D used	α (our result)	α (other estimates)
2	1.9	1.05	1.03 ± 0.07[a]
			1.11 ± 0.02[b]
			1.15 ± 0.03[c]
			1.1[d,e]
			1.132 ± 0.003[f]
3	2.45	1.28	1.34 ± 0.11[a]
			1.35 ± 0.04[c]
			1.33[e]
4	3	1.50	1.47 ± 0.11[a]
			1.58 ± 0.10[c]
5	3.5	1.75	1.70 ± 0.16[a]
>6	4	2	2.04 ± 0.04[a]
			2[c]
			2 (Bethe Lattice)

[a] [10].
[b] [11].
[c] [12] (through $\alpha = 1/\tilde{\nu}$).
[d] [9] (through $\alpha = D/d$).
[e] [4].
[f] [14] (through $\alpha = \nu_{\parallel}/\nu$).

In conclusion, our result appears to be roughly consistent with available data. Moreover, the elastic backbone [4], being the union of the shortest paths, must have higher dimensionality than α. Anyhow, as the loops in the elastic backbone are probably not very numerous, we may expect that this lower bound α is close to the exact value.

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